Exam. Code : 211003
Subject Code : 4974

M.Sc. Mathematics 3rd Sem. (Batch 2020-22)

NUMBER THEORY

Paper—MATH-586

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt any **FIVE** questions in all, selecting at least **ONE** question from each Section. The **fifth** question may be attempted from any Section. All questions carry equal marks.

SECTION—A

1. (a) Let p be a prime number. Then prove that $x^2 \equiv 1 \ (mod\ p)$ if and only if $x \equiv \pm 1 \ (mod\ p)$. 10

   (b) State and prove Chinese Remainder Theorem. 10

2. (a) Let q be a prime factor of $a^2 + b^2$. If $q \equiv 3 \pmod 4$ then prove that $q|a$ and $q|b$. 10

   (b) If p is a prime then prove that there exist $\varphi(\varphi(p^2)) = (p - 1)\ \varphi(p - 1)$ primitive roots modulo p^2. 10

SECTION—B

3. (a) If $p = 2^k + 1$ is prime, then prove that every quadratic nonresidue of p is a primitive root of p. 10

(b) If p is an odd prime, then prove that $\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)=0$ and there are precisely $(p-1)/2$ quadratic and $(p-1)/2$ quadratic non-residues of p. 10

4. (a) If $p \neq 3$ is an odd prime, then prove that

$$\left(\frac{3}{p}\right)=\begin{cases}1 & \text{if } p \equiv \pm 1 \pmod{12} \\ -1 & \text{if } p \equiv \pm 5 \pmod{12}\end{cases}.$$ 10

(b) If p and q are odd primes satisfying $p = q + 4a$ for some a, then prove that $\left(\frac{a}{p}\right)=\left(\frac{a}{q}\right)$. 10

SECTION—C

5. (a) Prove that for every positive integer n, $\Sigma_{d|n}\ \varphi(d) = n$. 10

(b) If $f(n) = \Sigma_{d|n}\ \mu(d)F(n/d)$ for every positive integer n, then prove that $F(n) = \Sigma_{d|n}\ f(d)$. 10

6. (a) Find all integers x and y such that $123x + 57y = 531$. 10

(b) Prove that the Diophantine equation $x^4 + y^4 = z^2$ has no solution in positive integers x, y, z. 10

SECTION—D

7. (a) If δ is real and irrational, then prove that there are infinitely many distinct rational numbers a/b such that $\left|\delta-\frac{a}{b}\right|<\frac{1}{b^2}$. 10

(b) Prove that $x^2 - dy^2 = -1$ has no solution if $d \equiv 3 \ (mod\ 4)$. 10

8. (a) Prove that the value of any infinite simple continued fraction $< a_0, a_1, a_2, ... >$ is irrational. 10

(b) Expand each of the following as infinite simple continued fractions :

$\sqrt{2}$, $\sqrt{2}-1$, $\frac{\sqrt{2}}{2}$, $\sqrt{3}$, $1/\sqrt{3}$. 10